UNCLASSIFIED

AD NUMBER
AD112382
LIMITATION CHANGES
TO: **Approved for public release; distribution is unlimited. Document partially illegible.**
FROM: **Distribution authorized to U.S. Gov't. agencies and their contractors; Administrative/Operational Use; MAR 1950. Other requests shall be referred to Air Force Research and Development Command, Project Rand, Washington, DC 20330. Document partially illegible.**
AUTHORITY
rand corp ltr via usaf 15 aug 1966

THIS PAGE IS UNCLASSIFIED

UNCLASSIFIED

AD 112 382

Reproduced

Armed Services Technical Information Agency

ARLINGTON HALL STATION; ARLINGTON 12 VIRGINIA

UNCLASSIFIED

FC

U.S. AIR FORCE
Project RAND
RESEARCH MEMORANDUM

This is a working paper. It may be expanded, modified, or withdrawn at any time. The views, conclusions, and recommendations expressed herein do not necessarily reflect the official views or policies of the U. S. Air Force.

The RAND Corporation
SANTA MONICA • CALIFORNIA

U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

A GENERAL PROBLEM IN THE CALCULUS
OF VARIATIONS WITH APPLICATIONS
TO PATHS OF LEAST TIME

M. R. Hestenes

RM-100

ASTIA Document Number AD 112382

1 March 1950

Assigned to ____________

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

46

A GENERAL PROBLEM IN THE CALCULUS OF VARIATIONS WITH APPLICATIONS TO PATHS OF LEAST TIME

M. R. Hestenes

Summary. The present paper is concerned with paths of least time for an airplane. This problem when formulated analytically leads us to a problem in the calculus of variations of a type which has not been adequately treated in the literature. However, the problem can be transformed into a problem, commonly called the problem of Bolza. The purpose of the present paper is to collect known results for the problem of Bolza and interpret them in terms of the new problem here formulated. This is done in Sections 2,3,4 and 5. In sections 10 and 11 we treat the case when additional constraints are imposed. Applications to paths of least time are found in Sections 8, 9 and 11.

The derivation of the results on the problem of Bolza which we have used can be found in a book by G. A. Bliss entitled "Lectures on the Calculus of Variations", The University of Chicago Press. The results given by Bliss are stated in a some-what different form than those here given. They are however equivalent and are related by the transformation given by Bliss in the introductory sections on the problem of Bolza.

1. <u>Introduction</u>. The problem at hand is that of finding an optimum path of an airplane P of known performance. In order to describe the equations of motion of the airplane we introduce the following notations:

$\underline{r}$ = position vector of the airplane
$\underline{v} = \dot{\underline{r}}$ = velocity of the airplane
$v = |\underline{v}|$ = speed of the airplane
$\underline{W}$ = weight vector of magnitude w
m = mass of the airplane
$\underline{T}$ = thrust of magnitude T
$\underline{L}$ = lift of magnitude L
$\underline{D}$ = drag of magnitude D
α = angle of attack
β = angle of bank
h = altitude
t = time

The equations of motion are then given by the equations

$$\frac{d}{dt}(m\,\underline{v}) = \underline{T} + \underline{L} + \underline{D} + \underline{W}$$

$$(1{:}1) \qquad \frac{dw}{dt} = \dot{w}(v,T,h)$$

In these equations L and D are known functions of α, β, h, v. It will be assumed that the thrust T is a prescribed function of v and h. The path of the airplane is then determined completely by initial values of $\underline{r}$, $\underline{v}$ and w and by the values of $\alpha(t)$ and $\beta(t)$ over the flight path. The problem at hand is to determine the functions

$$\alpha(t)\ ,\ \beta(t) \qquad\qquad 0 \leq t \leq t_2$$

which will minimize the time of flight t_2 among all paths with prescribed conditions on the initial and final values $\underline{r}(0)$, $\underline{v}(0)$, $w(0)$, $\underline{r}(t_2)$, $\underline{v}(t_2)$.

The problem just described suggests the following general problem in the calculus of variations. Consider a class of functions and a set of parameters

$$a_h(t) \quad \text{and} \quad b_\rho \qquad (h = 1,\ldots,m;\ \rho = 1,\ldots,r)$$

and a class of arcs

$$q_i(t) \qquad (t_1 \leqq t \leqq t_2;\ i = 1,\ldots,n)$$

connected by the differential equations

$$q_i' = \dot{Q}_i(t,q,a) \tag{1:2}$$

and end conditions

$$\begin{aligned} t_1 &= T_1(b) & q_i(t_1) &= Q_{i1}(b) \\ t_2 &= T_2(b) & q_i(t_2) &= Q_{i2}(b) \end{aligned} \tag{1:3}$$

We seek to find a set a_h, b_ρ, and q_i which minimizes a function of the form

$$I = g(b) + \int_{t_1}^{t_2} L(t,q,a)dt.$$

This problem, hereafter referred to as <u>problem</u> A, is one of Bolza type and is equivalent to the problem of Bolza under an assumption usually made in the development of its theory.

The problem described initially is a special case of problem A if we set $a_1(t) = \alpha(t)$, $a_2(t) = \beta(t)$, $b_1 = t_2$. The quantities $q_1,\ldots,q_7$ denote the components of $\underline{r}$ and $\underline{v}$ and w. The equations (1:1) are of the form (1:2). The equations (1:3) described the end values $t_1 = 0$, $q_i(t_1)$ = constant, $q_i(t_2)$ = constant, and $t_2 = b_1$. The function to be minimized is $I = g(b) = b_1$, the time of flight. Normally, $q_7(t_2) = w(t_2)$ is not prescribed.

2. <u>Problem of Bolza</u>. The problem formulated in the last section is a special case of a very general problem, commonly called the problem of Bolza. This problem has been formulated in many ways. The formulation we propose to use is the following one. Consider a class of elements

$$b_\rho \quad , \quad x_j(t) \qquad (t_1 \leqq t \leqq t_2;\ \rho = 1,\dots,r;\ j = 1,\dots,p)$$

made up of a set of numbers $(b_1,\dots,b_r)$ and a set of functions $x_1(t),\dots,x_p(t)$. Such a system will be called an <u>arc</u>. The first r components of this arc are constants. We shall be interested only in arcs that satisfy a system of differential equations

$$(2{:}1) \qquad \phi_i(t,x,x') = 0 \qquad\qquad (i = 1,\dots,n < p)$$

and end conditions

$$(2{:}2) \qquad t_1 = t_1(b) \quad , \quad x_i(t_1) = X_{i1}(b)$$

$$t_2 = t_2(b), \qquad x_i(t_2) = X_{i2}(b) \quad .$$

We seek in a class of arcs satisfying conditions (2:1) and (2:2), one which minimizes a function of the form

$$(2{:}3) \qquad I = g(b) + \int_{t_1}^{t_2} f(t,x,x')dt.$$

This problem is called the <u>problem of Bolza</u> and will be referred to as <u>problem B</u>. Observe that if there are no x's one has a minimum problem involving a function of r variables $b_1,\dots,b_r$. If there are no b's one has a fixed end point problem. We admit the case of $f \equiv 0$. This case is commonly called the <u>problem of Mayer</u>.

The general problem A formulated in the last section is reducible to one of type B. To this end we set

$$x_i(t) = q_i(t) \qquad (i = 1,\dots,n)$$

$$(2:4) \qquad x_{n+h}(t) = \int_{t_1}^{t} a_h(t)dt \qquad (h = 1,\dots,m)$$

so that $a_h = x'_{n+h}$. Set

$$\phi_i(t,x,x') = \dot{Q}_i(t,x_1,\dots,x_n,x'_{n+1},\dots,x'_{n+m}) - x'_i$$

$$(2:5) \qquad X_{i1}(b) = Q_{i1}(b) \quad , \quad X_{n+h,1}(b) = 0$$

$$X_{i2}(b) = Q_{i2}(b) \quad , \quad X_{n+h,2}(b) = b_{r+h},$$

where $b_{r+1},\dots,b_{r+m}$ are m additional constants to be adjoined to the set $b_1,\dots,b_r$ given in problem A. Thus, to every set of functions $a_h(t)$, b_ρ in problem A there corresponds a unique arc

$$b_1,\dots,b_{m+r} \; , \; x_1(t),\dots, x_{m+n}(t) \qquad (t_1 \leq t \leq t_2)$$

satisfying the differential equations

$$(2:6) \qquad \phi_i(t, x, x') = 0$$

and end conditions

$$t_1 = T(b) \; , \; x_j(t_1) = X_{j1}(b) \qquad (j = 1,\dots,m+n)$$

$$t_2 = T(b) \; , \; x_j(t_2) = X_{j2}(b) \; .$$

Moreover every arc of this type determines a unique set $b_1,\dots,b_r$, $a_h(t) = x'_{n+h}(t)$ satisfying the conditions in problem A. In terms of the new variables $b_1,\dots,b_{m+n}$, $x_1(t),\dots, x_{m+n}(t)$ the function to be minimised takes the form

$$I = g(b_1,\dots,b_r) + \int_{t_1}^{t_2} L(t,x_1,\dots,x_n,x'_{n+1},\dots,x'_{n+m})dt.$$

The problem of minimizing I subject to the conditions (2:6) and (2:7) is clearly equivalent to problem A.

Conversely problem B can be reduced to one of type A if one assumes that there exist $m = p - n$ functions $\phi_{n+h}(t,x,x')$ $(h = 1,\ldots,m)$ of class C'' such that the equations

$$\phi_i(t,x,x') = 0$$

$$\phi_{n+h}(t,x,x') = a_h$$

have unique solutions

$$x'_j = P_j(t,x,a) \qquad (j = 1,\ldots,p) \tag{2:8}$$

on the domain under consideration. The functions $x_j(t)$ are then completely determined when the values of $x_j(t_1)$ and $a_h(t)$ are known. Consequently if we eliminate the derivatives x'_j appearing in (2:3) problem B becomes equivalent to that of minimizing

$$I = g(b) + \int_{t_1}^{t_2} f(t,x,\ P(t,x,a))dt$$

in a class of arcs

$$a_h(t)\ ,\ b_\rho\ ,\ x_j(t) \qquad (t_1 \leq t \leq t_2)$$

satisfying the conditions (2:8) and (2:1). This problem is of type A with $q_j(t) = x_j(t)$.

The problem of Bolza as formulated above has been studied extensively in the literature. A comprehensive treatment of this problem can be found in a recent book by G. A. Bliss entitled "Lectures in the Calculus of Variations" published by the University of Chicago Press. In this book an extensive bibliography can be found for this problem. The formulation given by Bliss differs somewhat in detail with the one here given. However, Bliss indicates how to interpret his results for the case here

considered. The next section will be devoted to these interpretations.

3. Necessary conditions for a minimum for problem B.

Consider now the problem B formulated at the beginning of Section 2. In this problem the variables $b = (b_1, \ldots, b_r)$ are restricted to lie on an open set B and the element (t,x,x') to lie in an open set $\mathcal{R}$. In the sequel we shall restrict ourselves to elements of this type, called admissible elements, even though no explicit mention may be made of this fact. An arc C

$$C: \qquad b_\rho \qquad x_j(t) \qquad (t_1 \leq t \leq t_2; \rho = 1,\ldots,r; j = 1,\ldots,p)$$

will be called admissible if (1) the functions $x_i(t)$ are continuous and have piecewise continuous derivatives on $t_1 t_2$ and (2) its elements (b,t,x,x') are admissible. We shall be concerned only with admissible arcs. Consequently the adjective "admissible" usually will be omitted.

Consider now an arc C_0 that is a solution to our problem. We make the following assumptions regarding C_0.

1) The arc C_0 is of class C'', that is, the functions $x_j(t)$ defining C_0 have continuous first and second derivatives

2) The matrix $\| \phi_{ix'_j} \|$ has rank n on C_0

3) The arc C_0 is normal in the sense to be described later in this section. The abnormal case is highly singular and will not be descussed. Moreover it is not likely to occur in applications if the problem is properly formulated.

Under these hypotheses there exists a unique set of continuous multipliers $\lambda_i(t)$ such that if we set

$$F(t,x,x',\lambda) = f + \lambda_i\phi_i \qquad \text{(i summed)}$$

the conditions I, II, III, IV described below hold on C_0.

I. <u>The first necessary condition</u>. <u>Along C_0 the Euler-Lagrange equations</u>

$$(3:1) \qquad \frac{d}{dt}(F - x'_j F_{x'_j}) = F_t, \quad \frac{d}{dt} F_{x'_j} = F_{x_j}, \quad \phi_i = 0$$

<u>hold. Moreover, the end values of C_0 are such that the transversality condition</u>

$$(3:2) \qquad \Big[(F - x'_j F_{x'_j}) T_{s\rho} + F_{x'_j} X_{js\rho}\Big]_{s=1}^{s=2} + g_\rho = 0$$

<u>hold</u>, where the subscript ρ on $T_s(b)$, $X_{is}(b)$, $g(b)$ denotes the derivative with respect to b_ρ evaluated on C_0. The quantity in the brackets is to be evaluated at the final endpoint of C_0 when $s = 2$ and at the initial endpoint when $s = 1$.

II. <u>The necessary condition of Weierstrass</u>. <u>At each element</u> (t,x,x',λ) <u>on C_0 the inequality</u>

$$E(t,x,x',\lambda,X') \geqq 0$$

<u>holds for every admissible element</u> (t,x,X'), <u>where</u>

$$E = F(X') - F(x') - (X'_j - x'_j) F_{x'_j}(x') ,$$

<u>and the elements not exhibited are</u> (t,x,λ).

III. <u>The necessary condition of Clebsch</u> (<u>Legendre</u>). <u>At each element</u> (t,x,x',λ) <u>on</u> C_0 <u>the inequality</u>

$$F_{x'_j x'_k} \pi_j \pi_k \geqq 0$$

<u>holds for every solution</u> $(\pi) \neq (0)$ <u>of the equations</u>

$$\phi_{ix'_j} \pi_j = 0 \quad .$$

At this point it is convenient to introduce the concept of nonsingularity. The arc C_0 will be said to be <u>non singular</u> in case the determinant

$$\begin{vmatrix} F_{x'_j x'_k} & \phi_{ix'_j} \\ \phi_{hx'_k} & 0 \end{vmatrix} \qquad (i,h = 1,\ldots,n;\ j,k = 1,\ldots,p)$$

is different from zero. This condition insures the existence of a 2p-parameter family of solutions of the Euler Equations (3:1) having C_0 as one of its members.

It remains to describe condition IV. To this end we need to introduce the concept of variations. Consider therefore a one-parameter family of arcs

$$C_a: \quad b_\rho(a) \quad , \quad x_j(t,a) \qquad t_1(a) \leq t \leq t_2(a)$$

satisfying the conditions of our problem and containing C_0 for $a = 0$. Let the operator δ denote the derivative with respect to a evaluated at $a = 0$. The quantity

$$\gamma: \qquad \beta_\rho = \delta b_\rho \quad , \quad \xi_j = \delta x_j \qquad t_1 \leq t \leq t_2$$

is called the <u>variation</u> of the family along C_0. Inasmuch as

$$\phi_i(t, x(t,a), x'(t,a)) = 0$$

is an identity in a we have

$$\delta\phi_i = \phi_{ix_j}\,\delta x_j + \phi_{ix'_j}\,\delta x'_j = 0$$

Consequently the variation γ satisfies the equations of variation

$$\Phi_i(t,\xi,\xi') = \phi_{ix_j}\,\xi_j + \phi_{ix'_j}\,\xi'_j = 0 \tag{3:3}$$

of ϕ_i along C_0, the derivatives of ϕ_i being evaluated on C_0.

Turning now to the end conditions we observe that $t_s(a) = T_s(b(a))$ $(s = 1,2)$. Hence, the equations

$$x_j[T_s(b(a)),a] = X_{js}[b(a)] \qquad (s = 1,2)$$

hold identically in a. Operating by δ yields

$$x'_j(t_s)T_{s\rho}\delta b_\rho + \delta x_j(t_s) = X_{js\rho}\delta b_\rho$$

where, as before, the subscript ρ on T_s, X_{js} denotes the derivative with respect to b_ρ evaluated on C_0. Setting

(3:4) $$C_{js\rho} = X_{js\rho} - x'_j(t_s)T_{s\rho}$$

it is seen that the variation γ satisfies variational end conditions

(3:5) $$\xi_j(t_s) = C_{js\rho}\beta_\rho \qquad (s = 1,2).$$

We are now in position to define normality. A system of the form

$$\gamma: \qquad \beta_\rho, \quad \xi_j(t) \qquad (t_1 \leq t \leq t_2)$$

will be called a variation in case the functions $\xi_j(t)$ are continuous and have piecewise continuous derivatives on $t_1 t_2$. The arc C_0 will be said to be normal if there exist 2p variations

$$\gamma_\sigma: \qquad \beta_{\rho\sigma}, \quad \xi_{j\sigma}(t) \qquad (t_1 \leq t \leq t_2;\ \sigma = 1,\ldots,2p)$$

satisfying equations (3:3), no proper linear combination of which satisfies equations (3:5). Such a family has this latter property if and only if the determinant

$$\begin{vmatrix} \xi_{j\sigma}(t_1) - C_{j1\rho}\beta_{\rho\sigma} \\ \xi_{j\sigma}(t_2) - C_{j2\rho}\beta_{\rho\sigma} \end{vmatrix}$$

is different from zero. For a normal arc C_0 every variation γ satisfying equations (3:3) and (3:5) is related to a one parameter

family C_a in the sense described in the preceding paragraph. This may not be the case for abnormal problems.. In fact if C_0 is abnormal, it may be the only arc satisfying the conditions of our problem.

When the integral I is evaluated along the family C_a described above and is differentiated twice with respect to a at a = 0, one obtains the second variation $I_2(\gamma)$ of I along C_0. By means of suitable manipulations the second variation $I_2(\gamma)$ can be put in the form

$$I_2(\gamma) = B_{\rho\sigma}\beta_\rho\beta_\sigma + \int_{t_1}^{t_2} 2w(t,\xi,\xi')dt$$

where $\rho,\sigma = 1,\ldots,r$; $j,k = 1,\ldots,p$;

$$2w = F_{x_jx_k}\xi_j\xi_k + 2F_{x_jx'_k}\xi_j\xi'_k + F_{x'_jx'_k}\xi'_j\xi'_k ,$$

$$B_{\rho\sigma} = \Big[(F_t - x'_j F_{x_j})T_{s\rho}T_{s\sigma} + F_{x_j}(T_{s\rho}X_{js\sigma} + T_{s\sigma}X_{js\rho})$$

$$+ (F - x'_j F_{x'_j})T_{s\rho\sigma} + F_{x'_j}X_{js\rho\sigma}\Big]_{s=1}^{s=2} + g_{\rho\sigma}$$

the subscripts ρ,σ denoting derivatives with respect to b_ρ and b_σ evaluated on C_0.

The fourth necessary condition can now be stated as follows:

IV. <u>The second order condition</u>. <u>The second variation</u> $I_2(\gamma)$ <u>of</u> I <u>along</u> C_0 <u>is nonnegative for every variation</u> γ <u>satisfying equations</u> (3:3) <u>and</u> (3:5).

4. Necessary conditions for a minimum for problem A.

The results described in the last section can be used to obtain necessary conditions for a minimum for problem A. In problem A the parameters $b = (b_1, \ldots, b_r)$ are restricted to lie on an open set B and (t,q,a) to lie in an open set S. We admit only arcs

$$C: \qquad a_h(t), \quad b_\rho, \quad q_i(t) \qquad (t_1 \leq t \leq x_2)$$

with b in B, (t,q,a) in S, the functions $q_i'(t)$ being piecewise continuous.

Consider now a minimizing arc C_0 having the following two properties:

1) the functions $q_i'(t)$, $a_h(t)$ belonging to C_0 are continuous and possess continuous derivatives.

2) the arc C_0 is normal in the sense described below. Under these assumptions there exist unique multipliers $p_i(t)$ of class C' such that if we set

$$H(t,q,p,a) = p_i Q_i - L,$$

the arc C_0 satisfies the conditions I, II, III, IV described below.

I. <u>The first necessary condition</u>. <u>On C_0 the equations</u>

$$(4{:}1) \qquad q_i' = H_{p_i}, \quad p_i' = -H_{q_i}, \quad H_{a_h} = 0$$

<u>hold and hence also the equation</u>

$$(4{:}2) \qquad \frac{d}{dt} H = H_t .$$

<u>Moreover, the end values of C_0 are such that the transversality condition</u>

$$(4{:}3) \qquad \Big[-H T_{s\rho} + p_i Q_{is\rho} \Big]_{s=1}^{s=2} + g_\rho = 0$$

<u>holds.</u>

In order to establish this result we introduce new variables $x_j(t)$ $(j = 1,\ldots,n+m)$ defined by equations (2:4) together with additional end conditions

$$x_{n+h}(t_1) = 0, \quad x_{n+h}(t_2) = b_{r+h} \qquad (h = 1,\ldots,m)$$

Since $a_h = x'_{n+h}$ it will be convenient to use a_h and x'_{n+h} interchangeably. Similarly we shall use q_i in place of x_i when it is convenient to do so.

Set

$$F(t,x,x';p) = L(t,q,a) + p_i(q'_i - \dot{Q}_i(t,q,a)) .$$

It is clear that

$$F + H = p_i q'_i$$

Consequently

$$F_t = -H_t , \quad F_{x_i} = -H_{q_i} , \quad F_{x'_i} = p_i \qquad (i = 1,\ldots,n)$$

(4:4)
$$F_{x_{n+h}} = 0 , \quad F_{x'_{n+h}} = -H_{a_h} , \quad F - x'_i F_{x'_i} = -H$$

Using these facts the Euler Lagrange equations

$$\frac{d}{dx} F_{x'_i} = F_{x_i} , \quad \frac{d}{dx} F_{x'_{n+h}} = F_{x_{n+h}} , \quad x'_i = \dot{Q}_i$$

take the form

$$p'_i = -H_{q_i} , \quad H_{a_h} = \text{const.} , \quad q'_i = H_{p_i}$$

Using the part of the transversality condition (3:2) corresponding to the new parameters $b_{r+1},\ldots,b_{r+m}$ we find that

$$H_{a_h} = 0$$

at $t = t_2$ and hence on the whole interval $t_1 t_2$. The equations (4:1) accordingly hold. Inasmuch as

$$F - x'_j F_{x'_j} = F - x'_i F_{x'_i} - x'_{n+h} F_{x'_{n+h}} \tag{4:5}$$

$$= -H + a_h H_{a_h}$$

and $H_{a_h} = 0$ the condition (4:2) and (4:3) follow from the remaining conditions (3:1) and (3:2).

It is interesting to observe that in case the equations $q'_i = \phi_i$ are of the form

$$q'_i = a_i$$

so that a_i is but another symbol for q'_i, we have

$$H_{a_i} = p_i - L_{a_i} = p_i - L_{q'_i} \quad .$$

Consequently along a solution of equations (4:1), p_i are the cannonical variables and H coincides with the Hamiltonian function. It follows that the functions p_i and H here used can be considered to be a generalization of these quantities.

When the Weierstrass E-function for problem B is interpreted in terms of H for problem A by the use of (4:4) and (4:5) it takes the form

$$E(t,q,p,a,A) = -H(t,q,p,A) + H(t,q,p,a) + (A_h - a_h)\, H_{a_h}(t,q,p,a). \tag{4:6}$$

Inasmuch as $H_{a_h} = 0$ along C_0 we have

II. <u>The necessary condition of Weierstrass</u>. <u>Along</u> C_0 <u>the inequality</u>

$$H(t,q,p,A) \leq H(t,q,p,a)$$

<u>must hold for every admissible element</u> (t,q,A).

Thus, H has a maximum value with respect to a_h along a minimizing curve C_0.

In a similar manner we obtain

III. The necessary condition of Clebsch (Legendre). At each element (t,q,p,a) of C_0 the inequality

$$H_{a_h a_k}\pi_h\pi_k \leqq 0$$

must hold for every set $(\pi) \neq (0)$.

The condition of nonsingularity for problem B when interpreted for problem A yields the following condition. The arc C_0 is nonsingular in case the determinant

$$| H_{a_h a_k} |$$

is different from zero along C_0.

For problem A a system

$$\gamma : \quad \alpha_h(t) \quad , \quad \beta_\rho \quad , \quad \xi_i(t) \qquad (t_1 \leqq t \leqq t_2)$$

will be called a variation in case the functions $\xi_i(t)$ are continuous and $\xi_i'(t)$, $a_h(t)$ are piecewise continuous on t_1t_2. The analogues of equations (3:3) are

(4:7) $$\xi_i' = \dot{Q}_{iq_j}\,\xi_j + \dot{Q}_{ia_h}\alpha_h$$

and the analogue of (3:5) is

(4:8) $$\xi_i(t_s) = C_{is\rho}\,\beta_\rho \qquad (s = 1,2),$$

where, as in problem B,

(4:9) $$C_{is\rho} = Q_{is\rho} - q_i'(t_s)\,T_{s\rho}$$

evaluated on C_0.

The arc C_0 is normal if there exists a set of 2n-variations

$$\gamma_\mu : \alpha_{h\mu} \qquad \beta_{\rho\mu}\,, \quad \xi_{i\mu}(t), \qquad (\mu = 1,\ldots,2n)$$

satisfying equations (4:7), no proper linear combination of which satisfies equations (4:8).

The second variation $I_2(\gamma)$ of I along C_0 takes the form

$$I_2(\gamma) = B_{\rho\sigma}\alpha_\rho\alpha_\sigma - \int_{t_1}^{t_2} 2w(t,\xi,\alpha)dt$$

where

$$2w = H_{q_iq_j}\xi_i\xi_j + 2H_{q_ia_h}\xi_i\alpha_h + H_{a_ha_k}\alpha_h\alpha_k,$$

$$B_{\rho\sigma} = g_{\rho\sigma} - \Big[H\,T_{s\rho\sigma} - p_iQ_{is\rho\sigma} + (H_t - q_i'H_{q_i'})T_{s\rho}T_{s\sigma} + H_{q_i}(Q_{is\rho}T_{s\sigma} + Q_{is\sigma}T_{s\rho})\Big]_{s=1}^{s=2}.$$

As before the subscripts ρ,σ denote derivatives with respect to b_ρ and b_σ.

IV. The second order condition. The second variation $I_2(\gamma)$ of I along C_0 is nonnegative for every variation γ satisfying equations (4:7) and (4:8).

5. The Case $L \equiv 0$. In the applications with which we shall be concerned the integrand L is identically zero. In this case an important phenomenon occurs which we shall now explain. To this end we shall need the following.

Lemma 5:1 Suppose $L \equiv 0$ and let C_0 be an arc satisfying the Euler-Lagrange equations (4:1) with a set of multipliers p_i, then the relation

$$p_i(t)\xi_i(t) = \text{constant} \tag{5:1}$$

holds for every variation γ satisfying equations (4:7).

For if we multiply equations (4:7) by p_i we obtain the relation

$$p_i\xi_i' = H_{q_i}\xi_i + H_{a_h}\alpha_h = 0.$$

Using the Euler-Lagrange equations (4:1) this becomes

$$p_i\xi_i' + p_i'\xi_i = \frac{d}{dt}(p_i\xi_i) = 0$$

that is, $p_i\xi_i$ is a constant, as was to be proved.

The functions $p_i(t)$ being solutions of a system of linear equations

$$p_i' = -p_j \dot{Q}_{jq_i}$$

do not vanish simultaneously unless they are identically zero. The multipliers associated with a minimizing arc do not vanish identically. Hence equations (5:1) state that the end value $\xi_i(t_1)$ and $\xi_i(t_2)$ for a solution of the equations (4:7) are not independent, that is, one cannot assign the values $\xi_i(t_1)$ and $\xi_i(t_2)$ arbitrarily and expect to be able to find a solution of equations (4:7) passing through the points $(t_1, \xi_i(t_1))$, $(t_2, \xi(t_2))$. When this result is interpreted in terms of the original system of equations $q_i' = \dot{Q}_i(t,q,a)$ it means that we cannot, in general, expect to be able to find a solution of the equation $q_i' = \dot{Q}_i(t,q,a)$ passing through an arbitrarily chosen pair of points $(t_1, q(t_1))$ $(t_1, q(t_2))$ is a neighborhood of the end values of C_0.

In order to illustrate the phenomenon, consider the case when $n = 2$, $t = x$, $q_1 = y$, $q_2 = z$ and the differential equations takes the form

$$y' = a \quad , \quad z' = \sqrt{1 + a^2}$$

Then $z(x)$ measures the arc length of the curve $y(x)$ in (x,y) space. The equations (4:1) with $H = p_1 a + p_2 \sqrt{1 + a^2}$ take the form

$$y' = a, \; z' = \sqrt{1 + a^2}\, , \; p_1' = 0, \; p_2' = 0, \; p_1 + p_2 \frac{a}{\sqrt{1 + a^2}} = 0 \; .$$

Thus p_1 and p_2 are constants. Setting

$$\sin \theta = -p_1/p_2$$

these equations reduce to the system

$$z' = \sqrt{1 + y'^2} \quad , \quad \frac{y'}{\sqrt{1 + y'^2}} = \sin \theta$$

It follows that $y' = \tan \theta$ and hence that

$$y = x \tan \theta + d \quad , \quad z = x \sec \theta + e$$

are the solutions of the Euler-Lagrange equations. The solutions that pass through the origin are given by

$$y = x \tan \theta \quad , \quad z = x \sec \theta$$

and hence lie on the cone

$$z^2 = x^2 + y^2 .$$

The arcs satisfying the given equations $z' = \sqrt{1 + y'^2}$ and passing through the origin must lie interior to this cone. To see this we need only recall that z denotes the length of arc of the projection of the curve in the xy-plane. Consequently z must exceed or equal the distance $\sqrt{x^2 + y^2}$ of the point (x,y) to the origin, that is, $z \geqq \sqrt{x^2 + y^2}$, as was to be shown. Since a solution of the Euler equations lies on the cone there exists points, nearly (namely, those exterior to the cone) which cannot be joined to the origin by an arc satisfying the given differential equations $z' = \sqrt{1 + y'^2}$.

The situation we have just described is characteristic of problems in which the integrand L is identically zero.

6. <u>The aerodynamic equations of an airplane in terms of right-handed coordinate systems</u> . The differential equations of motion of an airplane are given by equations (1:1) in the introduction. We shall now select a suitable coordinate system to which the theory described in the preceding sections can be applied. It will be convenient to derive our equations vectorially, using right-handed coordinate systems. These results will then be interpreted in terms of the left-handed systems normally used. We restrict ourselves to short paths in which the curvature of the earth need not be taken into account.

As a reference frame we choose a right-handed system of x-, y-, z- axes, with z denoting the altitude of the point (x,y,z). Let ξ_0, η_0, ζ_0 denote unit vectors in the direction of the x-, y-, z- axes respectively. Let ξ be a unit vector in the direction of the velocity $\underline{v}$ of the airplane and let ζ be unit vector in the direction

of the lift. The right handed triple ξ, $\eta = \zeta \times \xi$, ζ will be used to describe the orientation of the airplane. The system ξ, η, ζ can be obtained from the system ξ_0, η_0, ζ_0 by three successive rotations, as follows:

First, a rotation about ζ_0 through an angle θ, called azimuth angle, to yield a system ξ_1, η_1, $\zeta_1 = \zeta_0$.

Second, a rotation about η_1 through an angle δ, called the angle of dive, to obtain a system ξ_2, $\eta_2 = \eta_1$, ζ_2 with $\xi_2 = \xi$.

Third, a rotation about $\xi_2 = \xi$ through an angle β, called the angle of bank, to yield the system ξ, η, ζ.

The vectors described in the Introduction are of the form

$$\underline{v} = v\,\xi\,,\quad \underline{D} = -D\xi\,,\quad \underline{L} = L\zeta\,,$$

(6:1)

$$\underline{T} = T(\xi\cos\alpha + \zeta\sin\alpha),\qquad \underline{w} = -w\zeta_0$$

where α is the angle of attack.

We are now in position to prove the following

Theorem 6:1 The functions $x(t)$, $y(t)$, $z(t)$ describing the position of the plane at the time t are connected with the speed $v(t)$, the azimuth angle $\theta(t)$, the angle of dive $\delta(t)$, the angle of bank $\beta(t)$, the angle of attack $\alpha(t)$, the weight $w(t)$ by the equations

$$\begin{aligned}
\dot{x} &= v\cos\theta\cos\delta\\
\dot{y} &= v\sin\theta\cos\delta\\
\dot{z} &= -v\sin\delta\\
\dot{v} &= \frac{g}{w}\left[T\cos\alpha - D - \frac{v}{g}\dot{W}\right] + g\sin\delta\\
\dot{\delta} &= \frac{-g}{vw}\cos\beta\left[T\sin\alpha + L\right] + \frac{g}{v}\cos\delta \qquad (6:2)\\
\dot{\theta} &= \frac{g}{vw}\sec\delta\,\sin\beta\left[T\sin\alpha + L\right]\\
\dot{w} &= \dot{W}
\end{aligned}$$

where T is the thrust, L is the lift, D is the drag, g is the gravitational acceleration, and $\dot{w}$ describes the rate of change of weight. The acceleration A_L of the plane in the direction of the lift is given by ..

(6:3) $$A_L = \frac{g}{W}[T \sin\alpha + L] - g \cos\delta \cos\beta$$

$$= -v\,(\dot{\theta} \cos\delta \sin\beta + \dot{\delta} \cos\beta)$$

The first three relations follow from the fact that the direction cosines of $\underline{v} = v\underline{\xi}$ with respect to the xyz—coordinate system are

(6:4) $$\underline{\xi}\cdot\underline{\xi}_0 = \cos\theta \cos\delta \;, \quad \underline{\xi}\cdot\underline{\eta}_0 = \sin\theta \cos\delta \;, \quad \underline{\xi}\cdot\underline{\zeta}_0 = -\sin\delta .$$

In order to derive formula for the acceleration $\underline{A} = \dot{\underline{v}}$ recall that

$$\underline{A} = \frac{d}{dt}(v\underline{\xi}) = \dot{v}\underline{\xi} + v\underline{\Omega} \times \underline{\xi}$$

where $\underline{\Omega}$ is the angular velocity of the reference frame $\underline{\xi}, \underline{\eta}, \underline{\zeta}$. The vector $\underline{\Omega}$ is the sum of three angular velocities

$$\underline{\Omega} = \dot{\beta}\,\underline{\xi} + \dot{\delta}\,\underline{\eta}_1 + \dot{\theta}\,\underline{\zeta}_0$$

by virtue of our choice of the angles β, δ, θ. It follows that the components A_ξ, A_η, A_ζ of $\underline{A}$ in the directions $\underline{\xi}$, $\underline{\eta}$, $\underline{\zeta}$ are

$$A_\xi = \underline{A}\cdot\underline{\xi} = \dot{v}$$
$$A_\eta = \underline{A}\cdot\underline{\eta} = v\underline{\Omega} \times \underline{\xi}\cdot\underline{\eta} = v\underline{\Omega}\cdot\underline{\zeta} = v\,\dot{\delta}\,\underline{\eta}_1\cdot\underline{\zeta} + v\,\dot{\theta}\,\underline{\zeta}_0\cdot\underline{\zeta}$$
$$A_\zeta = \underline{A}\cdot\underline{\zeta} = v\underline{\Omega} \times \underline{\xi}\cdot\underline{\zeta} = -v\underline{\Omega}\cdot\eta = -v\dot{\delta}\,\underline{\eta}_1\cdot\underline{\eta} - v\,\dot{\theta}\,\underline{\zeta}_0\cdot\underline{\eta}$$

Observing that

(6:5) $$\underline{\eta}\cdot\underline{\eta}_1 = \cos\beta \,, \qquad \underline{\eta}\cdot\underline{\zeta}_0 = \cos\delta \sin\beta$$
$$\underline{\zeta}\cdot\underline{\eta}_1 = -\sin\beta \,, \qquad \underline{\zeta}\cdot\underline{\zeta}_0 = \cos\delta \cos\beta$$

we find that

(6:6) $$A_\xi = \dot{v}$$
$$A_\eta = v(-\dot{\delta}\sin\beta + \dot{\theta}\cos\delta \cos\beta).$$
$$A_\zeta = -v(\dot{\delta}\cos\beta + \dot{\theta}\cos\delta \sin\beta).$$

Solving the last two equations for $\dot{\delta}$ and $\dot{\theta}$ we obtain

(6:7) $$\dot{\delta} = -\frac{1}{v}(A_\eta \sin\beta + A_\zeta \cos\beta),$$
$$\dot{\theta} = \frac{\sec\delta}{v}(A_\eta \cos\beta - A_\zeta \sin\beta).$$

Returning now to equation (1:1), that is, to the relation

$$\frac{d}{dt}\left(\frac{W}{g}\,\underline{v}\right) = \frac{W}{g}\,\underline{A} + \frac{\dot{W}}{g}\,\underline{v} = \underline{T} + \underline{D} + \underline{L} + \underline{W}$$

and making use of (6:1),(6:4) and (6:5) it is seen that the components A_ξ, A_η, A_ζ of $\underline{A}$ are also given by the formulas

$$A_\xi = \frac{g}{W}\left[T\cos\alpha - D - \frac{v}{g}\dot{W}\right] + g\sin\delta$$

(6:8) $$A_\eta = -g\cos\delta\,\sin\beta$$

$$A_\zeta = \frac{g}{W}\left[T\sin\alpha + L\right] - g\cos\delta\,\cos\beta\,.$$

Consequently

$$A_\eta\sin\beta + A_\zeta\cos\beta = \frac{g\cos\beta}{W}\left[T\sin\alpha + L\right] - g\cos\delta$$

$$A_\eta\cos\beta - A_\zeta\sin\beta = \frac{-g\sin\beta}{W}\left[T\sin\alpha + L\right]$$

Combining these results with (6:7) gives the formulas for $\dot{\delta}$ and $\dot{\theta}$ in (6:2). The formula for $\dot{v} = A_\xi$ in (6:2) is obtained from the first equation in (6:8). Equation (6:3) follows from (6:8) and (6:6).

7. <u>The aerodynamic equations of an airplane in terms of a left handed system.</u> The coordinate system used in the last section was chosen so that the rules in vector analysis could be applied directly. As a result the azimuth angle is measured in a counterclockwise as viewed by the pilot. In practice the azimuth angle is normally measured in the clockwise direction. We shall accordingly replace θ by $-\theta$ and shall reverse the y-axis so that a lefthanded system is obtained. We shall also replace the angle of dive δ by <u>the angle of climb</u> $\gamma = -\delta$. When these changes have been made Theorem 6:1 can be restated as follows:

Theorem 7:1 Let the position of the plane be described by a left handed coordinate system with the z-axis as the vertical axis. The functions $x(t)$, $y(t)$, $z(t)$ describing the position of the plane at any time t are connected with the speed $v(t)$, the azimuth angle $\theta(t)$, the angle of climb $\gamma(t)$, the angle of bank $\beta(t)$, the angle of attack $\alpha(t)$, the weight $w(t)$, by the differential equations

$$
(7{:}1)\qquad
\begin{aligned}
\dot{x} &= v\cos\theta\cos\gamma \\
\dot{y} &= v\sin\theta\cos\gamma \\
\dot{z} &= v\sin\gamma \\
\dot{v} &= \frac{g}{w}\left[T\cos\alpha - D - \frac{v}{g}\dot{W}\right] - g\sin\gamma \\
\dot{\gamma} &= \frac{g}{vw}\cos\beta\left[T\sin\alpha + L\right] - \frac{g}{v}\cos\gamma \\
\dot{\theta} &= \frac{g}{vw}\sec\gamma\sin\beta\left[T\sin\alpha + L\right] \\
\dot{w} &= \dot{W}
\end{aligned}
$$

where T is the thrust, L is the lift, D is the drag, g is the gravitational acceleration, and $\dot{W}$ describes the rate of change of weight. The acceleration A_L of the plane in the direction of the lift is given by

$$
(7{:}2)\qquad
\begin{aligned}
A_L &= \frac{g}{w}\left[T\sin\alpha + L\right] - g\cos\gamma\cos\beta \\
&= v\,(\dot{\theta}\cos\gamma\sin\beta + \dot{\gamma}\cos\beta)
\end{aligned}
$$

8. Paths of least time, unrestricted case. Consider now an airplane P with initial conditions

$$x = x_1,\ y = y_1,\ z = z_1,\ v = v_1,\ \gamma = \gamma_1,\ \theta = \theta_1,\ w = w_1,\ \text{at } t_1 = 0$$

to determine the path with terminal conditions

$$x = x_2,\ y = y_2,\ z = z_2,\ v = v_2,\ \gamma = \gamma_2,\ \theta = \theta_2\ \text{ at } t = t_2$$

traversed in the least time t_2. It is understood that the equations of motion are those described in the last section. This problem is of the type described in Section 4 if we set

$$a_1(t) = \alpha(t), \quad a_2(t) = \beta(t), \quad b_1 = T_2, \quad b_2 = w_2 = Q_{72}.$$

$$q_1(t) = x(t), \quad q_2(t) = y(t), \quad q_3(t) = z(t), \quad q_4(t) = v(t)$$

$$q_5(t) = \gamma(t), \quad q_6(t) = \theta(t), \quad q_7(t) = w, \quad g(b) = b_1, \quad L \equiv 0$$

If we denote the multipliers p_i introduced in section 4 by

$$p_x, \; p_y, \; p_z, \; p_v, \; p_\gamma, \; p_\theta, \; p_w$$

respectively, then the function H takes the form

$$\begin{aligned} H = {} & v(p_x \cos\theta \cos\gamma + p_y \sin\theta \cos\gamma + p_z \sin\gamma) \\ & -g\left(p_v \sin\gamma + \frac{p_\gamma}{v} \cos\gamma\right) \\ & +\frac{gp_v}{w}\left(T\cos\alpha - D - \frac{v}{g}\dot{W}\right) \\ & +\frac{g}{vw}(T\sin\alpha + L)(p_\gamma \cos\beta + p_\theta \sec\gamma \sin\beta) \\ & +p_w \dot{W} \end{aligned}$$

It is convenient to divide H into three parts

(8:1) $$H = F - G + p_w \dot{W},$$

where

$$\begin{aligned} F = {} & v(p_x \cos\theta \cos\gamma + p_y \sin\theta \cos\gamma + p_z \sin\gamma) \\ & -g\left(p_v \sin\gamma + \frac{p_\gamma}{v}\cos\gamma\right), \end{aligned}$$

(8:2) $$\begin{aligned} G = {} & \frac{gp_v}{w}\left[D - T\cos\alpha + \frac{v}{g}\dot{W}\right] \\ & -\frac{g}{vw}\left[T\sin\alpha + L\right](p_\gamma \cos\beta + p_\theta \sec\gamma \sin\beta). \end{aligned}$$

The function F is independent of the characteristics of the airplane and the function G depends upon these characteristics.

The system of equations (4:1) gives us in addition to equation (7:1) the equations

$$
\begin{aligned}
\dot{p}_x &= -H_x = 0 \\
\dot{p}_y &= -H_y = 0 \\
\dot{p}_z &= -H_z = G_z - p_w \dot{W}_z \\
\dot{p}_v &= -H_v = G_v - F_v - p_w \dot{W}_v \\
\dot{p}_\gamma &= -H_\gamma = G_\gamma - F_\gamma \quad . \\
\dot{p}_\theta &= -H_\theta = -F_\theta \\
\dot{p}_w &= -H_w = G_w \\
H_\alpha &= G_\alpha = 0 \\
H_\beta &= G_\beta = 0
\end{aligned}
\tag{8:3}
$$

The transversality conditions (4:3) with $b_1 = t_2$, $b_2 = w_2$, $g = b_1$ yields the relations

$$H = 1, \qquad p_w = 0 \qquad\qquad \text{at } t = t_2. \tag{8:4}$$

Since H is independent of t equation (4:2) tells us that H is a constant along the path. Consequently, by (8:1) and (8:4)

$$H = F - G + p_w \dot{W} = 1$$

or

$$p_w = \frac{1 - F + G}{\dot{W}} \tag{8:5}$$

The equations $\dot{p}_w = G_w$ may be replaced by equation (8:5).

We now turn to the equations $G_\alpha = G_\beta = 0$ in (8:3). In order to determine the consequences of these equations we assume that D and L are expressible in the form

$$L = L_1\alpha , \quad D = D_0 + {}^1\!/_2\, D_2\alpha^2$$

where L_1, D_0, D_2 are independent of α. Moreover we assume that α is so small, that we can replace $\sin\alpha$ by α and $\cos\alpha$ by 1. Then

$$G_\alpha = \frac{g}{vw}\left[p_v\, D_2\alpha v - (T + L_1)(p_\gamma \cos\beta + p_\theta \sec\gamma \sin\beta)\right] = 0.$$

Hence,

$$(8:6) \qquad \alpha = \frac{T + L_1}{vD_2}\left(\frac{p_\gamma}{p_v}\cos\beta + \frac{p_\theta}{p_v}\sec\gamma\sin\beta\right).$$

Similarly

$$G_\beta = -\frac{g}{vw}\left[T\alpha + L\right](-p_\gamma \sin\beta + p_\theta \sec\gamma\cos\beta) = 0.$$

Consequently

$$(8:7) \qquad \tan\beta = \frac{p_\theta}{p_\gamma}\sec\gamma.$$

In these derivations we have assumed that p_v and p_γ do not vanish. We shall now show that the Legendre condition requires them to be positive. The Legendre condition III of Section 4, with the equality excluded, states that

$$H_{\alpha\alpha}\, A^2 + 2H_{\alpha\beta}\, AB + H_{\beta\beta}\, B^2 < 0$$

unless $A = B = 0$. In terms of G this becomes

$$G_{\alpha\alpha}\, A^2 + 2G_{\alpha\beta}\, AB + G_{\beta\beta}\, B^2 > 0$$

unless $A = B = 0$. From the theory of quadratic forms this condition is equivalent to the inequalities

$$(8:8) \qquad G_{\alpha\alpha} > 0 \quad , \quad G_{\alpha\alpha}\, G_{\beta\beta} > G^2_{\alpha\beta}$$

Inasmuch as

$$G_{\alpha\alpha} = \frac{g}{w}\, p_v\, D_2$$

we see that p_v must be positive since D_2 is known to be positive.

Also

$$G_{\alpha\beta} = -\frac{g}{vw}\left[T + L_1\right](-p_\gamma \sin\beta + p_\theta \sec\gamma \cos\beta) = 0$$

by virtue of the equation preceding (8:7). Consequently, by (8:8)

$$G_{\beta\beta} = \frac{g}{vw}\left[T\alpha + L\right](p_\gamma \cos\beta + p_\theta \sec\gamma \sin\beta) > 0.$$

We have accordingly

$$-p_\gamma \sin\beta + p_\theta \sec\gamma \cos\beta = 0$$
$$p_\gamma \cos\beta + p_\theta \sec\gamma \sin\beta > 0$$

Inasmuch as $\cos\beta > 0$ these equations can hold simultaneously only in case $p_\gamma > 0$, as can be seen by multiplying the first relation by $-\sin\beta$ and the second by $\cos\beta$ and adding.

The results described in this section can be summarized as follows.

Theorem 8:1. If an airplane transverses a path of least time with end values prescribed as described above, there exist multipliers p_x, p_y, p_z, p_v, p_γ, p_θ such that the following relations hold with F and G defined by equations (8:2)

$$\dot{x} = v \cos\theta \cos\gamma$$
$$\dot{y} = v \sin\theta \cos\gamma$$
$$\dot{z} = v \sin\gamma$$
$$\dot{v} = \frac{g}{w}\left[T - D - \frac{v}{g}\dot{W}\right] - g \sin\gamma$$
$$\dot{\gamma} = \frac{g}{vw}\cos\beta\left[T\alpha + L\right] - \frac{g}{v}\cos\gamma$$
$$\dot{\theta} = \frac{g}{vw}\sec\gamma \sin\beta\left[T\alpha + L\right]$$
$$\dot{w} = \mathring{W}$$
$$\alpha = \frac{T+L_1}{vD_2}\left(\frac{p_\gamma}{p_v}\cos\beta + \frac{p_\theta}{p_v}\sec\gamma \sin\beta\right) =$$
$$\frac{T+L_1}{vD_2\,p_v}\left(p_\gamma^2 + p_\theta^2 \sec^2\gamma\right)^{1/2}$$

$$(8:9) \qquad \tan\beta = \frac{p_\theta}{p_\gamma}\sec\gamma$$

$$\dot p_z = G_z - (1-F+G)\frac{\dot W_z}{W}$$

$$\dot p_v = G_v - F_v - (1-F+G)\frac{\dot W_v}{W}$$

$$\dot p_\gamma = G_\gamma - F_\gamma$$

$$\dot p_\theta = -F_\theta$$

p_x and p_y are constants

$p_v > 0$, $p_\gamma > 0$ along the path.

$F - G = 1$ at $t = t_2$.

In the problem formulated above the terminal values of x, y, z, v, γ, θ were prescribed. If we did not prescribe one of these values, say x_2, then the problem would contain an additional parameter $b_3 = x_2$ and the transversality condition (4:3) would require that $p_x = 0$. Consequently, we have the following

Theorem 8:2. <u>If the problem described above is modified so as to not prescribe the terminal value of x then the multiplier $p_x = 0$ at $t = t_2$. A similar statement holds for the remaining variables y, z, v, γ, θ. If p_v or p_γ vanishes the problem becomes a singular, as can be seen from the formulas for α and β in</u> (8:9).

In view of the last remark it follows that the velocity v and the angle of climb γ must be prescribed at the end value.

9. <u>Motion of an airplane in a vertical plane</u>. If we assume that the airplane moves in a vertical plane, the angle of bank β is zero. By a suitable choice of axis we can suppose that $y = 0$ and that the x-axis have been chosen so that $\dot x$ is positive. In this case the equations of motion take the simpler form

$$\dot{x} = v \cos \gamma$$
$$\dot{z} = v \sin \gamma$$
$$\dot{v} = \frac{g}{w}\left[T \cos\alpha - D - \frac{v}{g}\dot{W}\right] - g \sin \gamma$$
$$(9:1)\qquad \dot{\gamma} = \frac{g}{vw}\left[T \sin\alpha + L\right] - \frac{g}{v} \cos \gamma$$
$$\dot{w} = \dot{W}$$

As in Section 8, set

$$F = v(p_x \cos \gamma + p_y \sin \gamma) - g(p_v \sin \gamma + \frac{p_\gamma}{v} \cos \gamma)$$
$$(9:2)\qquad G = \frac{g}{w} p_v \left[D - T \cos \alpha + \frac{v}{g}\dot{W}\right] - \frac{g}{vw} p_\gamma \left[T \sin\alpha + L\right]$$

The path of least time with prescribed initial values

$$x_1,\ z_1,\ v_1,\ \gamma_1,\ w_1, \quad \text{at } t = t_1 = 0$$

and terminal values

$$x_2,\ z_2,\ v_2,\ \gamma_2 \qquad \text{at } t = t_2$$

satisfies the conditions described in the following theorem, provided we replace $\cos\alpha$ by 1 and $\sin\alpha$ by α.

<u>Theorem</u> 9:1. <u>If an airplane traverses the path of least time</u> t_2 <u>with end values prescribed as described above, there exist multipliers</u> p_x, p_z, p_v, p_γ <u>such that the following relations hold</u>

$$\dot{x} = v \cos \gamma$$
$$\dot{z} = v \sin \gamma$$
$$\dot{v} = \frac{g}{w}\left[T - D - \frac{v}{g}\dot{W}\right] - g \sin \gamma$$
$$\dot{\gamma} = \frac{g}{vw}\left[T\alpha + L\right] - \frac{g}{v} \cos \gamma$$
$$\dot{w} = \dot{W}$$
$$(9:3)\qquad \alpha = \frac{p_\gamma}{p_v} \quad \frac{T + L_1}{D_2}$$

$$\dot{p}_z = G_z - (1 - F + G)\frac{\dot{W}_z}{\dot{W}}$$

$$\dot{p}_v = G_v - F_v - (1 - F + G)\frac{\dot{W}_v}{\dot{W}}$$

$$\dot{p}_\gamma = - F_\gamma$$

$$p_x = \text{constant}$$

$$p_v > 0, \; p_\gamma > 0$$

$$F - G = 1 \quad \text{at } t = t_2.$$

In these equations it is assumed that D and L are of the form

(9:4) $\qquad D = D_0 + \frac{1}{2}D_2\alpha^2 \; , \quad L = L_1\alpha$

where D_0, D_2, L_1 are independent of α. If the terminal value of x is not prescribed then $p_x = 0$.

This result can be obtained formally from (8:9) by setting $\beta = 0$, $p_y = p_\theta = 0$ and disregarding the equations involving $\dot{y}, \theta$, $\dot{p}_\theta$.

10. A problem with additional constraints. Consider now the case in which the arcs

C: $\qquad a_h(t), \quad b_\rho \quad , \quad q_i(t) \qquad (t_1 \leqq t \leqq t_2)$

not only satisfy a set of differential equations

(10:2) $\qquad q_i' = Q_i(t,q,a)$

but also a set of auxiliary conditions

(10:3) $\qquad \phi_\sigma(t,q,a) = 0. \qquad (\sigma = 1,\ldots,\ell \leqq m)$

As before we suppose that the end conditions are of the form

$$t_1 = T_1(b) \quad , \quad q_i(t_1) = Q_{i1}(b)$$

(10:4) $\qquad t_2 = T_2(b) \quad , \quad q_i(t_2) = Q_{i2}(b)$

and that the function to be minimized is of the form

$$I = g(b) + \int_{t_1}^{t_2} L(t,q,a)\,dt.$$

This problem will be called <u>problem A'</u>. It differs from problem A in that additional constraints (10:3) have been added.

Consider now a minimizing arc C_0 having the following three properties

1) the functions q_i', $a_h(t)$ belonging to C_0 are continuous and have piecewise continuous derivatives.

2) The matrix $\|\phi_{\sigma a_h}\|$ has rank ℓ along C_0.

3) The arc C_0 is normal in the sense described below.

Under these assumptions there exist continuous multipliers $p_i(t)$, $\lambda_\sigma(t)$ such that if we set

(10:5) $$H(t,q,p,a,\lambda) = p_i\dot{Q}_i - L - \lambda_\sigma \phi_\sigma$$

the arc C_0 satisfies conditions I, II, III, IV described below.

I. <u>The first necessary condition. On C_0 the equations</u>

(10:6) $$q_i' = H_{p_i}\ , \quad p_i' = -H_{q_i}\ , \quad H_{a_h} = 0\ , \quad \phi_\sigma = 0$$

<u>hold and hence also the equations</u>

(10:7) $$\frac{d}{dt} H = H_t$$

<u>Moreoever, the end values of C_0 are such that the transversality</u> condition

(10:8) $$\Big[-H\,T_{s\rho} + p_i\,Q_{is\rho}\Big]_{s=1}^{s=2} + g_\rho = 0$$

The proof of this result is like that of the analogous result given in §4 if we use the function

$$F(t,x,x',p,\lambda) = L(t,q,a) + \lambda_\sigma \phi_\sigma(t,q,a) + p_i\big[q_i' - \dot{Q}_i(t,q,a)\big]$$

in place of the function F defined in Section 4.

The Weierstrass E-function is of the form (4:6) and since $H_{a_h} = 0$ along C_0 we have

II. <u>The necessary condition of Weierstrass.</u> <u>Along</u> C_0 <u>the inequality</u>

$$H(t,q,p,A,\lambda) \leqq H(t,q,p,a,\lambda)$$

<u>must hold for every admissible element</u> (t,q,A) <u>satisfying the condition</u> $\phi_\sigma(t,q,A) = 0$.

Similarly, by virtue of condition III described in Section 3, we have

III. <u>The necessary condition of Clebsch (Legendre).</u> <u>At each element</u> (t,q,p,a,λ) <u>on</u> C_0 <u>the inequality</u>

$$H_{a_h a_k} \pi_h \pi_k \leqq 0$$

<u>must hold for every solution</u> $(\pi) \neq (0)$ <u>of the equations</u>

$$\phi_{\sigma a_h} \pi_h = 0$$

If we interpret the condition of nonsingularity for problem B in terms of problem A' we find that C_0 is nonsingular in case the determinant

$$\begin{vmatrix} H_{a_h a_k} & \phi_{\sigma a_h} \\ \phi_{\tau a_k} & 0 \end{vmatrix} \qquad (\sigma,\tau = 1,\ldots,\ell;\ h,k = 1,\ldots,m)$$

is different from zero along C_0.

For problem A' we are interested in variations

$$\gamma : \qquad \alpha_h(t) \ ,\ \beta_\rho \ ,\ \xi_i(t) \qquad (t_1 \leqq t \leqq t_2)$$

which satisfy the conditions

$$(10:9) \qquad \xi_i' = \overset{\circ}{Q}_{iq_j}\xi_i + \overset{\circ}{Q}_{ia_h}\alpha_h$$

$$0 = \phi_{\sigma q_i}\xi_i + \phi_{\sigma a_h}\alpha_h$$

together with the end conditions

$$(10:10) \qquad \xi_i(t_s) = C_{is\rho}\beta_\rho \qquad (s = 1,2;\ \rho = 1,\dots,r)$$

where $C_{is\rho}$ is given by (4:9). These equations differ from those given for problem A in that we have adjoined the equations of variation of the functions $\phi_\sigma(t,q,a)$ along C_0.

The arc C_0 is <u>normal</u> in case there exists 2n-variations

$$\gamma_\mu: \qquad \alpha_{h\mu}, \quad \beta_{\rho\mu}, \quad \xi_{i\mu}(t) \qquad (\mu = 1,\dots,2n)$$

satisfying equations (10:9), no proper linear combination of which satisfies (10:10).

The second variation $I_2(\gamma)$ takes the same form as for problem A, the function H being defined by (10:5). Consequently we have

IV. <u>The second order condition. The second variation $I_2(\gamma)$ of I along C_0 is nonnegative for every variation γ satisfying equations</u> (10:9) <u>and</u> (10:10).

11. <u>A problem with inequalities as constraints</u>. We now consider a modification of problem A in which we have inequalities as constraints. Consider therefore a class of arcs

$$C: \qquad a_h(t), \quad b_\rho, \quad q_i(t) \qquad (t_1 \leq t \leq t_2)$$

satisfying a set of differential equations

$$(11:1) \qquad q_i' = \dot{Q}_i(t,q,a)$$

together with a set of inequalities

$$(11:2) \qquad \phi_\sigma(t,q,a) \geqq 0 \qquad (\sigma = 1,\dots,\ell \leqq m)$$

in addition to a set of end conditions

$$(11:3) \qquad t_s = T_s(b), \quad q_i(t_s) = Q_{is}(b) \qquad (s = 1,2).$$

Again the function to be minimized is of the form

$$I = g(t \quad + \int_{t_1}^{t_2} L(t,q,a)dt.$$

This problem will be called problem A".

Consider now an arc C_0 that is a minimizing arc for problem A". We make the following assumptions regarding C_0.

(1) The functions $q_i'(t)$, $a_h(t)$ belonging to C_0 are continuous and have piecewise continuous derivatives.

(2) The matrix $\|\phi_{\sigma a_h}\|$ has rank ℓ at each point of C_0 at which $\phi_\sigma = 0$.

(3) The arc C_0 is normal in the sense described below.

Under these assumptions there exist continuous multipliers $p_i(t)$, $\lambda_\sigma(t)$ such that if we set

(11:4) $$H(t,q,p,a,\lambda) = p_i \dot{Q}_i - L - \lambda_\sigma \phi_\sigma$$

then conditions I, II, III, IV described below hold on C_0.

I. The first necessary condition. Along C_0 the equations

(11:5) $$q_i' = H_{p_i}\ , \quad p_i' = -H_{q_i}\ , \quad H_{a_h} = 0, \quad \phi_\sigma \geqq 0$$

must hold and hence also

(11:6) $$\frac{d}{dt} H = H_t\ .$$

Moreover, the multiplier $\lambda_\sigma(t)$ vanishes at each point of C_0 at which $\phi_\sigma > 0$. The end values of C_0 are such that the transversality conditions

(11:7) $$\left[-H\, T_{s\rho} + p_i\, Q_{is\rho}\right]_{s=1}^{s=2} + g_\rho = 0$$

hold.

The proof of this result can be made by a device used by Valentine, "The Problem of Lagrange with Differential Inequalities as Added Side Conditions", Contributions to the Calculus of Variations 1933-37, The University of Chicago Press. To this end we introduce new variables

$$x_i(t) = q_i(t) \qquad x_{n+h}(t) = \int_{t_1}^{t} a_h(t)\,dt ,$$

$$y_\sigma = \int_{t_1}^{t} \sqrt{\phi_\sigma}\,dt , \quad x_{n+h}(t_2) = b_{r+h}, \; y_\sigma(t_2) = b_{r+m+\sigma}.$$

Under this transformation our problem becomes that of finding in a class of arcs

$$b_\mu , \quad x_i(t) , \; y_\sigma(t) \quad (t_1 \leqq t \leqq t_2 ; \mu = 1,\ldots,r+m+\ell ; \; i = 1,\ldots,n+m ; \sigma = 1,\ldots,\ell)$$

satisfying the differential equations

$$x_i' - \mathring{Q}_i(t,x_1,\ldots,x_n,x_{n+1}',\ldots,x_{n+m}') = 0 \quad (i = 1,\ldots,n)$$

$$y_\sigma'^2 - \phi_\sigma(t,x_1,\ldots,x_n,x_{n+1}',\ldots,x_{n+m}') = 0$$

and end conditions

$$t_s = T_s(b_1,\ldots,b_r) , \quad x_i(t_s) = Q_{is}(b_1,\ldots,b_r) \quad (s = 1,2)$$

$$x_{n+h}(t_1) = 0 , \qquad x_{n+h}(t_2) = b_{r+h} \qquad (h = 1,\ldots,m)$$

$$y_\sigma(t_1) = 0 , \qquad y_\sigma(t_2) = b_{r+m+\sigma} \qquad (\sigma = 1,\ldots,\ell)$$

one which minimizes

$$I = g(b) + \int_{t_1}^{t_2} L(t,x_1,\ldots,x_n,x_{n+1}',\ldots,x_{n+1}',\ldots,x_{n+m}')\,dt$$

This problem is one of type B described in Section 3. Setting

$$F(t,x,x',y',p,\lambda) = L + \lambda_\sigma(\phi_\sigma - y_\sigma'^2) + p_i(x_i' - \mathring{Q}_i)$$

it is seen, as in Section 4, that the equations

$$\frac{d}{dt} F_{x_i'} = F_{x_i} , \quad x_i' = \mathring{Q}_i , \qquad y_\sigma'^2 = \phi_\sigma$$

are equivalent to the set (11:5). Since y_σ does not appear explicitly, the corresponding Euler equations tell us that $F_{y'_\sigma}$ is a constant along C_0. The transversality condition corresponding to $y_\sigma(t_2) = b_{r+m+\sigma}$ requires that this constant be zero. Hence

$$F_{y'_\sigma} = -2\lambda_\sigma y'_\sigma = 0 \qquad (\sigma \text{ not summed})$$

From this result we conclude that $\lambda_\sigma = 0$ whenever $y'_\sigma \neq 0$ that is, whenever $\phi_\sigma > 0$. The equations (11:7) are obtained by the argument given in Section 4.

Since $F_{y'_\sigma} = 0$ along C_0 the Weierstrass E-function takes the form (4:6). Hence we have

II. The necessary condition of Weierstrass. At each element (t,q,p,a,λ) on C_0, the inequality

$$H(t,q,p,A,\lambda) \leqq H(t,q,p,a,\lambda)$$

holds for every admissible element (t,q,A) for which $\phi_\sigma(t,q,A) \geqq 0$.

Similarly, by virtue of condition III for problem B, we have

III. The necessary condition of Clebsch (Legendre). At each element (t,q,p,a,λ) the inequality

$$H_{a_h a_k}\pi_h\pi_k + 2\lambda_\sigma \zeta_\sigma^2 \leqq 0$$

holds for all sets (π_k, ζ_σ) such that

$$\phi_{\sigma a_h}\pi_h = 2\zeta_\sigma\sqrt{\phi_\sigma} \qquad (\sigma = 1,\ldots,\ell)$$

This condition can be restated as follows.

III. The necessary condition of Clebsch (Legendre). The multipliers $\lambda_\sigma(t)$ satisfy the relation $\lambda_\sigma(t) \leqq 0$ along C_0, the equality holding whenever $\phi_\sigma > 0$. Moreover at each element (t,q,p,a,λ) on C_0 we have

$$H_{a_h a_k}\pi_h\pi_k \leqq 0$$

holding for all sets $(\pi) \neq (0)$ satisfying the relations

$$\phi_{\mu a_h}\pi_h = 0$$

where μ ranges over the integers $1,\ldots,\ell$ for which $\phi_\mu = 0$.

The arc C_0 will be said to be nonsingular in case at each element (t,q,p,a,λ) on C_0 the determinant

$$\begin{vmatrix} H_{a_h a_k} & \phi_{\mu a_h} \\ \phi_{\nu a_k} & 0 \end{vmatrix}$$

is different from zero where μ, ν ranges over the integers $1,\ldots,\ell$ for which $\phi_\mu = 0$ not only at (t,q,a) but also at neighboring elements (t,q,a) on C_0.

The variations

$$\gamma: \qquad \alpha_h(t), \quad \beta_\rho, \quad \xi_i(t) \qquad (t_1 \leqq t \leqq t_2)$$

have associated with them a set of conditions of the form

$$\text{(11:8)} \qquad \begin{aligned} \xi_i' &= \dot{Q}_{iq_j}\xi_j + \dot{Q}_{ia_h}\alpha_h \\ \Phi_\sigma &= \phi_{\sigma q_j}\xi_j + \phi_{\sigma a_h} = 0 \text{ whenever } \phi_\sigma = 0 \end{aligned}$$

together with the end conditions

$$\text{(11:9)} \qquad \xi_i(t_s) = C_{ish}\beta_h \qquad (s = 1,2)$$

The new condition in (11:8) can be derived as follows. Let be the variations of the variables y_σ described above. The equations of variations of the equations

$$\phi_\sigma = y_\sigma'^2$$

are

$$\text{(11:10)} \qquad \Phi_\sigma = 2y_\sigma'\eta_\sigma' = 2\eta_\sigma'\sqrt{\phi_\sigma} \quad (\sigma = 1,\ldots,\ell).$$

Consequently, $\Phi_\sigma = 0$ whenever $\phi_\sigma = 0$. At points, where $\phi_\sigma > 0$ the values of η_σ' can be chosen so that (11:10) holds.

The arc C_0 is normal in case there exists 2n variations

$$\alpha_{h\mu}(t), \quad \beta_{\rho\mu}, \quad \xi_{i\mu}(t) \qquad (t_1 \leqq t \leqq t_2;\ \mu = 1,\ldots,2n)$$

satisfying equations (11:8), no proper linear combination of which

satisfies equations (11:9).

Let $I_2(\gamma)$ be defined as in Section 4, with H defined by (11:4). Adjoining the variations η_σ to γ it is found by applying condition IV for problem B to the problem described in the derivation of condition I above that the inequality

$$I_2(\gamma) - \int_{t_1}^{t_2} 2\lambda_\sigma \eta_\sigma'^2 \, dt \geqq 0$$

holds for all variations

$$\alpha_h(t) \quad , \quad \beta_\rho \quad , \quad \xi_i(t) \quad , \quad \eta_\sigma(t)$$

satisfying equations (11:8), (11:10) and (11:9). Recall that

(1) $\lambda_\sigma \leqq 0$, (2) $\lambda_\sigma = 0$ whenever $\phi_\sigma > 0$, (3) η_σ' is arbitrary when $\phi_\sigma = 0$. In view of this fact we have

IV. <u>The second order condition</u>. <u>The second variations</u> $I_2(\gamma)$ <u>along</u> C_0 <u>must satisfy the condition</u> $I_2(\gamma) \geqq 0$ <u>for every variation</u> γ <u>satisfying equations</u> (11:8) <u>and</u> (11:9).

12. <u>Paths of least time with a condition on the lift coefficient</u>. In section 8 we discussed the optimum path of an airplane in which we tacitly assumed that the variables remained in the domain of validity of the differential equations. We shall now consider a restriction encountered due to the fact that the coefficient of lift C_L cannot exceed a critical value C_L^*, that is,

$$C_L(M,\alpha) \leqq C_L^*(M).$$

Here M is the mach number and α is the angle of attack. Since we have assumed that C_L contains α as a linear factor this condition is equivalent to one of the form

$$\alpha \leqq A(M) = A(v,z) \tag{12:1}$$

The mach number M is a function of the speed v and the altitude. If we adjoin the condition (12:1) to the problem described in Section 8 we obtain one of the type described in the last section with

$$\phi(v, z, \alpha) = A(v, z) - \alpha$$

The results stated in Theorem 8:1 must be modified as follows:

Theorem 12:2 *If an airplane traverses a path of least time subject to the condition* (12:1), *there exist multipliers* p_x, p_y, p_z, p_v, p_γ, p_θ *such that the following relations hold with* F and G *defined by equations* (8:2)

$$
\begin{aligned}
\dot{x} &= v \cos\theta \cos\gamma \\
\dot{y} &= v \sin\theta \cos\gamma \\
\dot{z} &= v \sin\gamma \\
\dot{v} &= \frac{g}{W}\left[T - D - \frac{v}{g}\dot{W}\right] - g \sin\gamma \\
\dot{\gamma} &= \frac{g}{vW} \operatorname{Cos}\beta \left[T\alpha + L\right] - \frac{g}{v} \cos\gamma \\
\dot{\theta} &= \frac{g}{vW} \sec\gamma \sin\beta \left[T\alpha + L\right]
\end{aligned}
$$

(12:2)

$$
\begin{aligned}
\dot{w} &= \dot{W} \\
\alpha &\leqq A(v,z) \\
\alpha &= \min\left[\frac{T + L_1}{v\, D_2}\left(\frac{p_v}{p_v} + \frac{p_\theta}{p_v} \sec\gamma \sin\beta\right) \text{ and } A(v,z)\right] \\
\tan\beta &= \frac{p_\theta}{p_\gamma} \sec\gamma \\
\dot{p}_z &= G_z + \lambda A_z - (1 - F + G)\frac{\dot{W}_z}{\dot{W}} \\
\dot{p}_v &= G_v + \lambda A_v - F_v - (1 - F + G)\frac{\dot{W}_v}{\dot{W}} \\
\dot{p}_\gamma &= G_\gamma - F_\gamma \\
\dot{p}_\theta &= F_\theta
\end{aligned}
$$

p_x and p_y are constants

$$
\begin{aligned}
&\lambda = 0 \quad \text{if} \quad \alpha < A(v,z) \\
&\lambda = G_\alpha \leqq 0 \quad \text{if} \quad \alpha = A(v,z) \\
&p_v > 0, \quad \text{if} \quad \alpha < A(v,z) \\
&p_\gamma > 0 \\
&F - G = 1 \quad \text{at } t = t_2.
\end{aligned}
$$

In this theorem it is understood that the initial and terminal conditions of the path are the same as those imposed in Theorem 8:1.

According to the results described in the last section the new multiplier λ is zero whenever $\alpha < A(v,z)$. It follows that the conditions described in Theorem 8:1 are valid in this case. When $\alpha = A(v,z)$ then by virtue of the theory developed in the last section we replace the function H given by (8:1) by

$$H^* = H - \lambda(A - \alpha).$$

Consequently

$$p_z = -H^*_z = G_z + \lambda A_z - (1-F+G)\frac{\dot{W}_z}{\dot{W}}$$

$$p_v = -H^*_v = G_v + \lambda A_v - F_v - (1-F+G)\frac{\dot{W}_v}{\dot{W}}$$

where, as before, p_w has been evaluated by the formula

$$H^* = F - G + p_w\dot{W} - \lambda(A-\alpha) = F - G + p_w\dot{W} = 1$$

From the relation

$$H^*_\alpha = H_\alpha + \lambda = -G_\alpha + \lambda = 0$$

we conclude that

$$\lambda = G_\alpha$$

whenever $\alpha = A(v,z)$. In fact this holds when $\alpha < A(v,z)$ since then $\lambda = G_\alpha = 0$. In view of condition III given in the last section we see that

$$\lambda = G_\alpha \leqq 0$$

along the minimizing arc C_0. When $\alpha = A(y,z)$ condition III states further that

$$H^*_{\alpha\alpha}\,\pi^2 + 2H^*_{\alpha\beta}\,\pi k + H^*_{\beta\beta}\,k^2 \leqq 0$$

whenever (π,k) satisfies the equation

$$\phi_\alpha \pi = -\pi = 0$$

where $\phi = A - \alpha$. Consequently $H^*_{\beta\beta} = -G_{\beta\beta} \leqq 0$.

By an argument like that made in the paragraph preceding Theorem 8:1 it is seen that $p_\gamma \geqq 0$ and in fact that $p_\gamma > 0$ if our problem is to be nonsingular. This proves the theorem.

13. <u>An illustrative example</u>. In order to illustrate the method described in Section 11 consider the following example. To find among all arcs

$$y(x) \qquad\qquad x_1 \leqq x \leqq x_2$$

joining the points

$$(x_1, y_1) = (-2\sqrt{2},\ 4) \quad , \quad (x_2,\ y_2) = (2\sqrt{2},\ 4)$$

of length $4 + \pi$ and having

$$y'^2 \leqq 1 \ ,$$

one which minimizes the area integral

$$\int_{x_1}^{t_2} y \ dx$$

This problem, as it stands, is not of the type described in Section 11. However by the introduction of new variables it can be reduced to one of this type. The first variable we shall introduce is

$$z(x) = \int_{x'}^{x} \sqrt{1 + y'^2} \ dx$$

which measures the length of arc from x_1 to x. The second variable, θ , is the inclination of the arc y(x), that is,

$$\tan \theta = y' \ .$$

The problem then becomes that of choosing

$$y(x), \ z(x) \text{ and } \theta(x)$$

such that

$$y' = \tan \theta$$

$$z' = \sec \theta$$

$$\text{(13:1)} \qquad x_1 = -2\sqrt{2} \quad , \quad y(x_1) = 4 \quad , \quad z(x_1) = 0$$

$$x_2 = 2\sqrt{2} \quad , \quad y(x_2) = 4 \quad , \quad z(x_2) = \pi + 4$$

$$\pi^2/16 \ - \theta^2 \geqq 0$$

$$\int_{x_1}^{x^2} y \ \ dx = \text{minimum}$$

This new problem is of the type described in section 11 if we set

$$t = x, \quad q_1 = y, \quad q_2 = z, \quad a = \theta .$$

The variables $b_1, \ldots, b_r$ of Section 11 are absent. The equations $\phi_\sigma \geqq 0$ reduce to

$$\phi = \frac{\pi^2}{16} - \theta^2 \geqq 0$$

The function H of Section 11 takes the form

$$H = p_y \tan\theta + p_z \sec\theta - y - \lambda\left(\frac{\pi^2}{16} - \theta^2\right)$$

Consequently condition I becomes

$$(13:2) \qquad \begin{aligned} y' &= \tan\theta \\ z' &= \sec\theta \\ p_y' &= 1 \\ p_z' &= 0 \\ H_\theta &= p_y \sec^2\theta + p_z \sec\theta \tan\theta + 2\lambda\theta = 0 \\ \lambda\left(\frac{\pi^2}{16} - \theta^2\right) &= 0 \\ \theta^2 &\leqq \pi^2/16 \end{aligned}$$

Moreover since H is independent of x we have

$$H = -b = \text{constant}.$$

From these relations it is seen that

$$p_y = x - a, \quad p_z = -r$$

where a and r are constants. Inasmuch as $H_\theta = 0$ and $H = -b$ we have

$$(13:3) \qquad \begin{aligned} (x-a)\sec^2\theta - r\sec\theta\tan\theta + 2\lambda\theta &= 0 \\ y = b + (x-a)\tan\theta - r\sec\theta & \end{aligned}$$

Consider now the case when

$$\theta^2 < \pi^2/16$$

Then $\lambda = 0$ and by (13:3)

$$(13:4) \qquad \begin{aligned} x &= a + r\sin\theta \\ y &= b - r\cos\theta \end{aligned}$$

Consequently, the arc must be a circular arc with its center at (a,b) and radius $\pm$ r. We shall see presently that r is positive

or zero. The corresponding arc length is

(13:5) $$z = r\theta + c$$

as can be seen from the relations

$$\frac{dz}{d\theta} = \frac{dz}{dx}\,\frac{dx}{d\theta} = r\ .$$

To show that r is positive we use the Legendre condition $H_{\theta\theta} \leqq 0$. This yields, by (13:4),

$$\begin{aligned} H_{\theta\theta} &= (x-a)\ 2\sec^2\theta\ \tan\theta - r\ (\sec^3\theta + \sec^2\theta\ \tan\theta\ ) \\ &= -r\ \sec\theta \leqq 0. \end{aligned}$$

Hence $r \geqq 0$, as was to be proved. Inasmuch as $|\sin\theta| \leqq 1/\sqrt{2}$ when $\theta^2 < \pi^2/_{16}$ we have the inequality

$$a - r/\sqrt{2} \leqq x \leqq a + r/\sqrt{2}$$

along the circular arc under consideration.

When $\theta = -\frac{\pi}{4}$ it follows from (13:3) and (13:2) that

$$\begin{aligned} y &= b - (x-a) - r\sqrt{2} \\ z &= \sqrt{2}\ (x-a) + c_1 \\ \lambda &= \frac{4}{\pi}\ (r/\sqrt{2} + x - a) \end{aligned}$$

Observe that

$$\lambda = 0 \text{ when } x = a - r/\sqrt{2}$$

The Legendre condition states that $\lambda \leqq 0$ it follows that

$$x \leqq a - r/\sqrt{2}$$

along that portion of the arc on which $\theta = -\frac{\pi}{4}$.

Similarly when $\theta = \frac{\pi}{4}$

$$\begin{aligned} y &= b + x - a - r\sqrt{2} \\ z &= \sqrt{2}\ (x-a) + c_2 \\ \lambda &= \frac{4}{\pi}(r/\sqrt{2} - x + a) \end{aligned}$$

Consequently

$$\lambda = 0 \text{ when } x = a + r/\sqrt{2}$$

Since $\lambda \leqq 0$ we see that

$$x \geqq a + r/\sqrt{2}$$

along that portion of the arc on which $\theta = \pi/4$.

From these considerations it is seen that the solution to our problem must be a segment of an arc defined by the relations

(13:6)
$$y = b - r\sqrt{2} - x + a \qquad x \leqq a - r/\sqrt{2}$$
$$y = b - \sqrt{r^2 - (x-a)^2} \qquad a - r/\sqrt{2} \leqq x \leqq a + r/\sqrt{2}$$
$$y = b - r\sqrt{2} + x - a \qquad a + r/\sqrt{2} \leqq x$$

The corresponding values of z are given by

(13:7)
$$z = (x-a)\sqrt{2} + r(1 - \frac{\pi}{4}) + c \qquad x \leqq a - r/\sqrt{2}$$
$$z = r \sin^{-1}\left(\frac{x-a}{r}\right) + c \qquad a - r/\sqrt{2} \leqq x \leqq a + r/\sqrt{2}$$
$$z = (x-a)\sqrt{2} - r(1 - \frac{\pi}{4}) + c \qquad a + r/\sqrt{2} \leqq x$$

The multiplier λ is given by

$$\lambda = \frac{4}{\pi}(r/\sqrt{2} + x - a) \qquad x \leqq a - r/\sqrt{2}$$
$$\lambda = 0 \qquad a - r/\sqrt{2} \leqq x \leqq a + r/\sqrt{2}$$
$$\lambda = \frac{4}{\pi}(r/\sqrt{2} - x + a) \qquad a + r/\sqrt{2} \leqq x$$

If we choose

$$a = 0, \quad b = 4, \quad r = 2, \quad c = 2 + \pi/2$$

the end conditions in (13:1) will be satisfied. For in this event we have

(13:8)
$$y = 4 - 2\sqrt{2} - x \qquad -2\sqrt{2} \leqq x \leqq -\sqrt{2}$$
$$y = 4 - \sqrt{4 - x^2} \qquad -\sqrt{2} \leqq x \leqq \sqrt{2}$$
$$y = 4 - 2\sqrt{2} + x \qquad \sqrt{2} \leqq x \leqq 2\sqrt{2}$$

and

$$z = x\sqrt{2} + 4 \qquad -2\sqrt{2} \leqq x \leqq -\sqrt{2}$$
$$z = 2 \arcsin \frac{x}{2} + 2 + \pi/2 \qquad -\sqrt{2} \leqq x \leqq \sqrt{2}$$
$$z = x\sqrt{2} + \pi \qquad \sqrt{2} \leqq x \leqq 2\sqrt{2}$$

In this event

$$\lambda = \frac{4}{\pi}(\sqrt{2} + x) \qquad -2\sqrt{2} \leqq x \leqq -\sqrt{2}$$

$$\lambda = 0 \qquad -\sqrt{2} \leqq x \leqq \sqrt{2}$$

$$\lambda = \frac{4}{\pi}(\sqrt{2} - x) \qquad \sqrt{2} \leqq x \leqq 2\sqrt{2}.$$

The multiplier λ is a continuous function of x whose derivative is discontinuous at $x = \pm\sqrt{2}$.

The variable θ which plays a role analogous to the angle of attack α in Section 12 is given by the relations

$$\theta = -\frac{\pi}{4} \qquad -2\sqrt{2} < x < -\sqrt{2}$$

$$\theta = 2 \arcsin \frac{x}{2} \qquad -\sqrt{2} \leqq x \leqq \sqrt{2}$$

$$\theta = \frac{\pi}{4} \qquad \sqrt{2} \leqq x \leqq 2\sqrt{2}$$

and is accordingly continuous. Its derivative with respect to x,

$$\theta' = 0 \qquad -2\sqrt{2} < x < -\sqrt{2}$$

$$\theta' = \frac{2}{\sqrt{4 - x^2}} \qquad -\sqrt{2} \leqq x \leqq \sqrt{2}$$

$$\theta' = 0 \qquad \sqrt{2} \leqq x \leqq 2\sqrt{2},$$

is discontinuous at $x = \pm\sqrt{2}$.

The derivatives $y' = \tan\theta$ and $z' = \sec\theta$ are continuous along the arc.

To show that the arc (13:8) is the only arc (13:6) of length $\pi + 4$ passing through the points $(\pm 2\sqrt{2}, 4)$, observe first that $a = 0$ since the value of y is the same at $x = 2\sqrt{2}$ as at $x = -2\sqrt{2}$. If $\frac{r}{\sqrt{2}} \geqq 2\sqrt{2}$ (i.e. $r \geqq 4$) the arc would be composed of a circular arc whose length could not exceed 2π. Hence $r < 4$ and the arc is made up of two line segments and a circular arc. Since $z = 0$ at $x = -2\sqrt{2}$ and $z = \pi + 4$ at $x = 2\sqrt{2}$ the equation (13:7) yield the relations

$$0 = -4 + r\left(1 - \frac{\pi}{4}\right) + c$$

$$\pi + 4 = 4 - r\left(1 - \frac{\pi}{4}\right) + c$$

Consequently

$$c = 2 + \pi/2 \quad , \quad r = 2$$

Using the fact that $y = 4$ when $x = 2\sqrt{2}$ we find from (13:6) that $b = 4$. The arc (13:8) is accordingly the only arc of the form (13:6) that satisfies the conditions of our problem.